# Test 2 Computational Methods of Science/Computational Mechanics, January 2022

Duration: 2 hours.
In front of the questions, one finds the points. The sum of the points plus 1 gives the end mark for this test. Criteria used for the grading are: demonstration of understanding, logical reasoning, correct use of terminology, correctness of results. The acronyms **Math** and **ME** mean for mathematics and mechanical engineering students, respectively.

1. Consider on the interval [0,1] the differential equation

$$\frac{du}{dx} - \frac{d}{dx}(\exp(x)\frac{du}{dx}) = 0 \tag{1}$$

with boundary conditions $u(0) = 1$ and $\frac{du}{dx}(1) = 2$.

(a) (1.7 points) Make a finite volume discretization of this equation on an equidistant grid with mesh size $h$ and where each boundary condition is applied at the interface of a control volume.

(b) (0.9 points) **Math**: Show that the bilinear form attached to the discrete problem in the previous part is given by

$$a(\mathbf{v}, \mathbf{u}) = \frac{1}{h}\{(2v_1u_1/h + v_nu_n - \sum_{j=1}^{n-1}(v_{j+1} - v_j)\left[(u_{j+1} + u_j)/2 + (u_{j+1} - u_j)\exp(x_{j+\frac{1}{2}})/h\right]\},$$

where the elements of the vector $\mathbf{u}$ are $u_1, u_2, ...., u_n$, similar for $\mathbf{v}$. You may end up with a slightly different expression for $a(\mathbf{v}, \mathbf{u})$ depending on choices you made in part a.

(c) (0.8 points) **Math**: Show that for the bilinear form given in the previous part $a(\mathbf{u}, \mathbf{u})$ is nonnegative. Hint: use $(a + b)(a - b) = a^2 - b^2$. Next show that there is no nonzero function that will make $a(\mathbf{u}, \mathbf{u}) = 0$, hence it is positive definite.

(b) (0.8 points) **ME**: Show that the discretization of part a will lead to a monotonous solution. If you where not able to find that discretization use $(u_{j+1} - u_{j-1})/4 - (1 + ((j + \frac{1}{2})h)^2)(u_{j+1} - u_j) + (1 + ((j - \frac{1}{2})h)^2)(u_j - u_{j-1}) = 0$.

(c) (0.9 points) **ME**: Show that the discretization derived in part a, divided by $h$, is a second-order accurate approximation of the given differential equation away from the boundaries. If you were not able to derive a discretization there, you may also use the one in the previous part, but then, you also have to determine *to which* differential equation it converges (since the discretization given in part b is not a discretization of differential equation (1)).

**Exam text continues at other side**

2. Consider the wave equation on the unit square, i.e. $[0,1]\times[0,1]$:

$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}).$$

for $t > 0$. The boundary conditions are $u(0,y,t) = g(t)$, $u_x(1,y,t) = 0$, and $u_y(x,0,t) = u_y(x,1,t) = 0$. Furthermore, we have the initial conditions $u(x,y,0) = 0$ and $u_t(x,y,0) = 0$.

(a) (1.7 points) Give the difference equations that occur by a finite difference discretization in space on an equidistant grid of this initial boundary value problem. The Dirichlet boundary condition will be applied at grid points and the Neumann conditions in the middle between two grid points. The mesh size in x- and y-direction may differ. Use subscripts $j$ and $k$ to number in $x$- and $y$-direction, respectively, to prepare for the next part. Also give the initial conditions. Next sketch how these equations are put into matrix vector form:

$$\frac{d^2}{dt^2}\mathbf{u} = A\mathbf{u} + \mathbf{b}(t) \tag{2}$$

by indicating what goes in $\mathbf{u}$, $A$ and $\mathbf{b}$, respectively.

(b) (1.2 points) Use the difference/Fourier method to get an estimate of the eigenvalues of $A$. These eigenvalues will be real. Hint: the two dimensional Fourier component is given by $\exp(i(j\theta + k\phi))$. Here $i$ is the imaginary unit and $\theta$ and $\phi$ can take any value in $[-\pi,\pi]$. Hint: You may need the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$.

(c) (0.8 points) Transform the system of second-order differential equations in (2) to a system of first-order differential equations. Don't forget the initial conditions!

(d) (0.3 points) If $\lambda$ is an eigenvalue of the matrix $A$ then $\mu = \pm\sqrt{\lambda}$ is an eigenvalue of the system in the previous part. Compute $\mu$ if you use for $\lambda$ the Fourier eigenvalues found in part b. If you were not able to find $\lambda$ you may assume it is $-(\sin\theta/\Delta x)^2 - (\sin\phi/\Delta y)^2$ where $\Delta x$ and $\Delta y$ are the mesh widths in $x$- and $y$-direction, respectively.

(e) (0.8 points) The classical Runge-Kutta method contains the interval [-2.7i,2,7i] in its region of absolute stability. Express the maximum allowed time step $\Delta t$ in terms of $\Delta x$ and $\Delta y$ when we apply it to the first-order system in part c.

(f) (0.5 points) Consider the eigenfrequency mode $\mathbf{u}(t) = \mathbf{v}\exp(i2\pi f t)$. Substitute it in (2) with $\mathbf{b}(t) \equiv 0$ and determine how frequency $f$ is related to an eigenvalue of $A$.

(g) (0.6 points) How can one use the power method to find the frequency nearest to a target frequency $f_\tau$?